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Space/time non-commutative field theories and causality

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Abstract. As argued previously, amplitudes of quantum field theories on non-commutative space and time cannot be computed using naïve path integral Feynman rules. One of the proposals is to use the Gell-Mann–Low formula with time-ordering applied before performing the integrations. We point out that the previously given prescription should rather be regarded as an interaction-point time-ordering. Causality is explicitly violated inside the region of interaction. It is nevertheless a consistent procedure, which seems to be related to the interaction picture of quantum mechanics. In this framework we compute the one-loop self-energy for a space/time non-commutative ϕ^4 theory. Although in all intermediate steps only three-momenta play a rôle, the final result is manifestly Lorentz covariant and agrees with the naïve calculation. Deriving the Feynman rules for general graphs, we show, however, that such a picture holds for tadpole lines only.

1 Introduction

Quantum field theories on non-commutative spaces are full of surprises, indicating that a true *understanding* of quantum field theory is still missing [1]. This means, on the other hand, that studying the quantization of field theories on non-commutative spaces we resolve the degeneracy of various methods developed for commutative geometries: The outcomes of different quantization schemes ported to non-commutative geometries will no longer coincide.

At the moment we know of two major challenges. First, the evaluation of Feynman graphs as a perturbative solution of the path integral produces a completely new type of infrared-like singularities [2,3] in non-planar graphs. This can be understood from the power-counting theorem [4] for non-commutative (massive, Euclidian) field theories, which implies the existence of two types (rings and commutants) of non-local divergences.

Second, the case of a Minkowskian signature of the non-commutative geometry ("space/time non-commutativity") turns out to be involved. It was pointed out in [5] that in the Minkowskian (non-degenerate) case the Wick rotation of an Euclidian Green's function does *not* give a meaningful result, first of all because unitarity would be lost [6]. The reason is that the Osterwalder–Schrader theorem [7] does not hold. Already in [8] there was given a proposal for a correct quantization of field theories on space/time non-commutative geometries: Starting with interaction Hamiltonians on a Fock space

$$H_{\mathrm{I}}(t) = \int_{x^0=t} \mathrm{d}^3x \; :(\phi\star\phi\star\cdots\star\phi)(x): \qquad (1)$$

(and averaging over the non-commutativity parameter) the contributions to the scattering amplitudes were defined as the Dyson series

$$G_n(x_1,\dots,x_k) := \frac{(-\mathrm{i})^n}{n!}\int \mathrm{d}t_1\dots\mathrm{d}t_n \times \langle 0\Big|T\phi(x_1)\dots\phi(x_k)H_{\mathrm{I}}(t_1)\cdots H_{\mathrm{I}}(t_n)\Big|0\rangle\,, \qquad (2)$$

where T denotes the time-ordering with respect to $\{x_1^0,\dots,x_k^0,t_1,\dots,t_n\}$ and $|0\rangle$ the vacuum state. Unitarity is preserved. In [5] there was added a second proposal, the iterative solution of the (interacting) field equation (Yang–Feldman approach), which has the advantage of being manifestly covariant. Unitarity is preserved as well. We refer, in particular, to [9].

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A third approach, the direct application of the Gell-Mann–Low formula for Green's functions,

$$G_n(x_1,\dots,x_k) := \frac{\mathrm{i}^n}{n!}\int \mathrm{d}^4z_1\dots\mathrm{d}^4z_n \times \langle 0\Big|T\phi(x_1)\dots\phi(x_k)\mathcal{L}_{\mathrm{I}}(z_1)\cdots\mathcal{L}_{\mathrm{I}}(z_n)\Big|0\rangle^{\mathrm{con}}\;, \tag{3}$$

where $\mathcal{L}_{\mathrm{I}}$ is the interaction Lagrangian, was elaborated in [10]. The superscript $^{\mathrm{con}}$ means projection onto the connected part. Unitarity was investigated in [11]. That approach was called "time-ordered perturbation theory" in [10], a name which we find ambiguous. The time-ordering in [10] is considered for external vertices and *interaction points* only, and not with respect to the *actual time-order of the fields* in the interaction Lagrangian. We give in Sect. 2 a few comments on the two natural ways of time-ordering. The version used in [10] is an *interaction-point time-ordering* (IPTO); it is explicitly acausal, and to be distinguished from a true *causal time-ordering*.

Explicit calculations for the proposed quantization schemes of space/time non-commutative field theories are technically much more cumbersome than Feynman graph computations. It is therefore desirable to extract a powerful calculus out of the general schemes. In a first step one has to familiarize oneself with the computational methods of the new approach.

For that purpose we compute in this paper the one-loop two-point function for a ϕ^4 theory on non-commutative space-time. The result of the indeed very lengthy but straightforward calculation in interaction-point time-ordered perturbation theory agrees with the naïve path integral computation of the relevant Feynman graph. Deriving in Sect. 4 the Feynman rules for IPTO, we show, however, that this is true for tadpole lines only (which should be removed anyway by normal ordering).

We may speculate that taking the true causal time-ordering in the Gell-Mann–Low formula one ends up with the usual Feynman rules involving the causal Feynman propagator. It seems, therefore, that causality and unitarity are mutually exclusive properties of space/time non-commutative geometries.

2 Comments on time-ordering and causality

By "non-commutative $\mathbb{R}^4$" one understands the *algebra* $\mathbb{R}^4_\theta$ of Schwartz class functions on ordinary four-dimensional space, equipped with the multiplication rule

$$(f\star g)(x)=\int \mathrm{d}^4s\int\frac{\mathrm{d}^4l}{(2\pi)^4}\,f\left(x-\tfrac{1}{2}\tilde{l}\right)\,g(x+s)\,\mathrm{e}^{\mathrm{i}ls}\;, \tag{4}$$

where $\tilde{l}^\nu := l_\mu\theta^{\mu\nu}$. The product (4) characterized by a real skew-symmetric translation-invariant tensor $\theta^{\mu\nu}=-\theta^{\nu\mu}$ of dimension $[\mathrm{length}]^2$ is associative and non-commutative; it is a *non-local* product on rapidly decreasing functions.

We consider a scalar field theory on $\mathbb{R}^4_\theta$ given by the classical action

$$\Sigma = \int \mathrm{d}^4z\,\Big(\tfrac{1}{2}g^{\mu\nu}(\partial_\mu\phi\star\partial_\nu\phi)(z)-\tfrac{1}{2}m^2(\phi\star\phi)(z) + \frac{g}{4!}(\phi\star\phi\star\phi\star\phi)(z)\Big)\;, \tag{5}$$

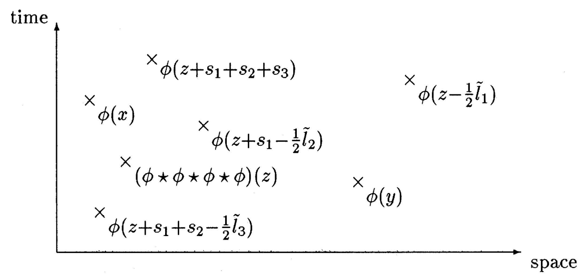
with $\phi\in\mathbb{R}^4_\theta$. By definition (4) we have

$$\begin{aligned}&\big(\phi\star\phi\star\phi\star\phi\big)(z)\\ &=\int\prod_{i=1}^{3}\left(\mathrm{d}^4s_i\frac{\mathrm{d}^4l_i}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}l_is_i}\right)\phi\left(z-\tfrac{1}{2}\tilde{l}_1\right)\phi\left(z+s_1-\tfrac{1}{2}\tilde{l}_2\right)\\ &\quad\times\phi\left(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3\right)\phi(z+s_1+s_2+s_3)\;.\end{aligned} \tag{6}$$

If $g^{\mu\nu}$ is the Minkowskian metric $g^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, we cannot simply Wick-rotate the Euclidian Green's functions obtained by evaluation of the path integral, see [5]. Here we shall follow the proposal of [10] and use the Gell-Mann–Low formula (3) to define the quantum field theory. However, one has to be more careful with the definition of the time-ordering. Let us consider the simplest case of the two-point function at first order in g,

$$G(x,y)=\frac{g}{4!}\int\mathrm{d}^4z\,\langle 0\Big|T\big(\phi(x)\phi(y)\big(\phi\star\phi\star\phi\star\phi\big)(z)\big)\Big|0\rangle\;. \tag{7}$$

(We put the missing factor i directly into the formula for the element of the S-matrix.) In the same manner as on commutative space-time, the integration over the interaction point is performed *after* taking the time-ordered product. Since the $\star$-product for $\theta^{0i}\neq 0$ is non-local in time, one has to say clearly what one understands under time-ordering. Let us discuss this nuance for the geometrical situation relevant for (7):



. (8)

This arrangement of fields corresponds to the following non-vanishing contribution to the true time-ordering of (7):

$$\begin{aligned}&G_{(8)}(x,y)\\ &=\int\mathrm{d}^4z\int\prod_{i=1}^{3}\left(\mathrm{d}^4s_i\frac{\mathrm{d}^4l_i}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}l_is_i}\right)\tau\left(s_1^0+s_2^0+s_3^0+\tfrac{1}{2}\tilde{l}_1^0\right)\\ &\quad\times\tau\left(z^0-\tfrac{1}{2}\tilde{l}_1^0-x^0\right)\tau\left(x^0-z^0-s_1^0+\tfrac{1}{2}\tilde{l}_2^0\right)\\ &\quad\times\tau\left(z^0+s_1^0-\tfrac{1}{2}\tilde{l}_2^0-y^0\right)\tau\left(y^0-z^0-s_1^0-s_2^0+\tfrac{1}{2}\tilde{l}_3^0\right)\\ &\quad\times\langle 0\Big|\phi(z+s_1+s_2+s_3)\phi\left(z-\tfrac{1}{2}\tilde{l}_1\right)\phi(x)\\ &\qquad\times\phi\left(z+s_1-\tfrac{1}{2}\tilde{l}_2\right)\phi(y)\phi\left(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3\right)\Big|0\rangle\;.\end{aligned} \tag{9}$$

Here, $\tau(t)$ denotes the step function $\tau(t)=1$ for $t>0$ and $\tau(t)=0$ for $t<0$. There are $6!=720$ different contributions to (7) when interpreting the time-ordering in the

Gell-Mann–Low formula as the name suggests. The time-ordering guarantees that causal processes only contribute to the S-matrix. Positive energy solutions propagate forward in time and negative energy solutions backward.

There exists a modification of (7), where the time-ordering is defined with respect to the *interaction point*:

$$\begin{aligned} &G'_{(8)}(x,y) \\ &= \int \mathrm{d}^4z \int \prod_{i=1}^{3} \left(\mathrm{d}^4 s_i \frac{\mathrm{d}^4 l_i}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i} l_i s_i} \right) \tau(x^0-z^0)\tau(z^0-y^0) \\ &\qquad \times \langle 0 \Big| \phi(x)\phi\left(z-\tfrac{1}{2}\tilde{l}_1\right) \phi\left(z+s_1-\tfrac{1}{2}\tilde{l}_2\right) \\ &\qquad \times \phi\left(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3\right) \phi(z+s_1+s_2+s_3)\phi(y) \Big| 0 \rangle \,. \end{aligned} \tag{10}$$

There are now only $3! = 6$ different contributions of this type. Since the individual fields are now (in most of the cases) at the wrong place with respect to the time-order, the interpretation (10) of the Gell-Mann–Low formula violates causality. Now both energy solutions propagate in any direction of time. There is, however, an argument in favor of (10): Contributions (2) to the Dyson series are precisely ordered with respect to the time stamp of the interaction Hamiltonians. It does not matter how the time-dependence of the interaction Hamiltonian is produced from the time-dependence of the constituents.

Since it is completely unclear how to *derive* the Gell-Mann–Low formula in the non-commutative setting, we have no guidance so far whether (9) or (10) (or none of the two) is the correct one. The authors of [10] do not mention (9). They use the exponential form of the $\star$-product, which is a formal translation[1] of a correct formula in momentum space, but which might be dangerous in position space. See also the discussion in [12]. Apart from avoiding subtleties with generalized derivatives, the use of (6) instead of the exponential form simplifies the calculations considerably.

3 The one-loop two-point function in "interaction-point time-ordered perturbation theory"

Since the calculation of the sum of terms (10) is (at least) by a factor of 120 simpler than the calculation of the sum of terms (9), we evaluate in this paper the one-loop two-point function interpreted according to (10). The name "time-ordered perturbation theory" used in [10] does not seem appropriate to us, because the previous discussion shows that this approach is precisely *not* based on time-ordering. We should better call it "interaction-point time-ordered perturbation theory", and use the symbol T_I instead of the true causal time-ordering T. The calculation can be shortened considerably when starting directly from the Feynman rule (39) derived in Sect. 4. But without computing at least one example one has little understanding for the starting point (34) of the general derivation.

With these remarks, the entire contribution to the one-loop two-point function in non-commutative ϕ^4 theory reads

$$\begin{aligned} &G(x,y) \\ &= \frac{g}{4!} \int \mathrm{d}^4z\, \langle 0 \Big| T_\mathrm{I}\big(\phi(x)\phi(y)\big(\phi\star\phi\star\phi\star\phi\big)(z)\big) \Big| 0 \rangle \\ &= \frac{g}{4!} \int \mathrm{d}^4z \Big(\tau(x^0-y^0)\tau(y^0-z^0) \\ &\qquad \times \big\langle 0 \big| \phi(x)\phi(y)\big(\phi\star\phi\star\phi\star\phi\big)(z) \big| 0 \big\rangle \\ &\quad + \tau(x^0-z^0)\tau(z^0-y^0) \\ &\qquad \times \big\langle 0 \big| \phi(x)\big(\phi\star\phi\star\phi\star\phi\big)(z)\phi(y) \big| 0 \big\rangle \\ &\quad + \tau(y^0-x^0)\tau(x^0-z^0) \\ &\qquad \times \big\langle 0 \big| \phi(y)\phi(x)\big(\phi\star\phi\star\phi\star\phi\big)(z) \big| 0 \big\rangle \\ &\quad + \tau(y^0-z^0)\tau(z^0-x^0) \\ &\qquad \times \big\langle 0 \big| \phi(y)\big(\phi\star\phi\star\phi\star\phi\big)(z)\phi(x) \big| 0 \big\rangle \\ &\quad + \tau(z^0-x^0)\tau(x^0-y^0) \\ &\qquad \times \big\langle 0 \big| \big(\phi\star\phi\star\phi\star\phi\big)(z)\phi(x)\phi(y) \big| 0 \big\rangle \\ &\quad + \tau(z^0-y^0)\tau(y^0-x^0) \\ &\qquad \times \big\langle 0 \big| \big(\phi\star\phi\star\phi\star\phi\big)(z)\phi(y)\phi(x) \big| 0 \big\rangle \Big) \,, \end{aligned} \tag{11}$$

with the $\star$-product given in (6). We follow the usual strategy to obtain in the end the amputated on-shell momentum-space one-loop two-point function. We insert (6) into (11) and split each field (at given position x) $\phi(x) = \phi^+(x) + \phi^-(x)$ into negative and positive frequency parts, which have the property

$$\phi^-(x)\big|0\big\rangle = 0 \,, \quad \big\langle 0\big|\phi^+(x) = 0 \,. \tag{12}$$

Our conventions are listed in the appendix; they are opposite to [10]. It is convenient now to commute the ϕ^- to the right and the ϕ^+ to the left, using the commutation rule

$$[\phi^-(x_1), \phi^+(x_2)] = D^+(x_1-x_2) \,, \tag{13}$$

where $D^+(x_1-x_2)$ is the positive frequency propagator

$$D^+(x_1-x_2) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\, \mathrm{e}^{-\mathrm{i}k^+(x_1-x_2)} \,, \tag{14}$$

with $\omega_k = \sqrt{\vec{k}^2+m^2}$, and $k^+_\mu = (+\omega_k, -\vec{k})$ is the positive energy on-shell four-momentum. A lengthy but completely standard computation yields

$$G(x,y) = G^{\mathrm{con}}(x,y) + G^{\mathrm{discon}}(x,y) \,, \tag{15}$$

$$G^{\mathrm{discon}}(x,y)$$

[1] The derivatives in the exponential form of the $\star$-product are generalized derivatives in the sense of distribution theory, not ordinary derivatives. As such one cannot apply the naïve rules of differential calculus. To make this transparent, write $\phi(x+a)\phi(y) = \exp(a^\mu\partial^x_\mu)\phi(x)\phi(y)$, and hide the exponential of the derivatives in the definition of the product. It would be completely wrong to use the step function $\tau(x^0-y^0)$ or $\tau(y^0-x^0)$ for the product $\phi(x+a)\phi(y)$. One of the authors (R.W.) is grateful to Edwin Langmann for explaining this matter to him

$$
\begin{aligned}
&= \frac{g}{4!}\int \mathrm{d}^4 z \int \prod_{i=1}^{3}\left(\mathrm{d}^4 s_i \frac{\mathrm{d}^4 l_i}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i} l_i s_i}\right)\\
&\quad \times\Big\{\Big(\tau(x^0-y^0)\tau(y^0-z^0)D^+(x-y)\\
&\qquad +\tau(x^0-z^0)\tau(z^0-y^0)D^+(x-y)\\
&\qquad +\tau(z^0-x^0)\tau(x^0-y^0)D(x-y)\Big)+(x\leftrightarrow y)\Big\}\\
&\quad \times\Big(D^+\left(-\tfrac12\tilde l_2-s_2+\tfrac12\tilde l_3\right)D^+\left(-\tfrac12\tilde l_1-s_1-s_2-s_3\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_1-s_1-s_2+\tfrac12\tilde l_3\right)D^+\left(-\tfrac12\tilde l_2-s_2-s_3\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_1-s_1+\tfrac12\tilde l_2\right)D^+\left(-\tfrac12\tilde l_3-s_3\right)\Big)\,, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&G^{\mathrm{con}}(x,y)\\
&= \frac{g}{4!}\int \mathrm{d}^4 z \int \prod_{i=1}^{3}\left(\mathrm{d}^4 s_i \frac{\mathrm{d}^4 l_i}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i} l_i s_i}\right)\\
&\quad\times\Big\{\Big(\tau(x^0-y^0)\tau(y^0-z^0)\\
&\qquad\times\Big\{\Big(D^+\left(-\tfrac12\tilde l_1-s_1-s_2-s_3\right)D^+\left(x-z-s_1+\tfrac12\tilde l_2\right)\\
&\qquad\qquad\times D^+\left(y-z-s_1-s_2+\tfrac12\tilde l_3\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_1-s_1-s_2+\tfrac12\tilde l_3\right)D^+\left(x-z-s_1+\tfrac12\tilde l_2\right)\\
&\qquad\qquad\times D^+(y-z-s_1-s_2-s_3)\\
&\qquad +D^+\left(-\tfrac12\tilde l_1-s_1+\tfrac12\tilde l_2\right)D^+\left(x-z-s_1-s_2+\tfrac12\tilde l_3\right)\\
&\qquad\qquad\times D^+(y-z-s_1-s_2-s_3)\\
&\qquad +D^+\left(-\tfrac12\tilde l_2-s_2-s_3\right)D^+\left(x-z+\tfrac12\tilde l_1\right)\\
&\qquad\qquad\times D^+\left(y-z-s_1-s_2+\tfrac12\tilde l_3\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_2-s_2+\tfrac12\tilde l_3\right)D^+\left(x-z+\tfrac12\tilde l_1\right)\\
&\qquad\qquad\times D^+(y-z-s_1-s_2-s_3)\\
&\qquad +D^+\left(-\tfrac12\tilde l_3-s_3\right)D^+\left(x-z+\tfrac12\tilde l_1\right)\\
&\qquad\qquad\times D^+\left(y-z-s_1-\tfrac12\tilde l_2\right)\Big)\\
&\quad +(x\leftrightarrow y)\Big\}\\
&\quad +\tau(x^0-z^0)\tau(z^0-y^0)\\
&\quad\times\Big\{D^+\left(-\tfrac12\tilde l_1-s_1-s_2-s_3\right)D^+\left(x-z-s_1-s_2+\tfrac12\tilde l_3\right)\\
&\qquad\qquad\times D^+\left(z+s_1-\tfrac12\tilde l_2-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_2-s_2-s_3\right)D^+\left(x-z-s_1-s_2+\tfrac12\tilde l_3\right)\\
&\qquad\qquad\times D^+\left(z-\tfrac12\tilde l_1-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_1-s_1-s_2+\tfrac12\tilde l_3\right)D^+(x-z-s_1-s_2-s_3)\\
&\qquad\qquad\times D^+\left(z+s_1-\tfrac12\tilde l_2-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_2-s_2+\tfrac12\tilde l_3\right)D^+(x-z-s_1-s_2-s_3)\\
&\qquad\qquad\times D^+\left(z-\tfrac12\tilde l_1-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_1-s_1-s_2-s_3\right)D^+\left(x-z-s_1+\tfrac12\tilde l_2\right)\\
&\qquad\qquad\times D^+\left(z+s_1+s_2-\tfrac12\tilde l_3-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_3-s_3\right)D^+\left(x-z-s_1+\tfrac12\tilde l_2\right)\\
&\qquad\qquad\times D^+\left(z-\tfrac12\tilde l_1-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_1-s_1+\tfrac12\tilde l_2\right)D^+(x-z-s_1-s_2-s_3)\\
&\qquad\qquad\times D^+\left(z+s_1+s_2-\tfrac12\tilde l_3-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_1-s_1-s_2+\tfrac12\tilde l_3\right)D^+\left(x-z-s_1+\tfrac12\tilde l_2\right)\\
&\qquad\qquad\times D^+(z+s_1+s_2+s_3-y)\\
&\qquad +D^+\left(-\tfrac12\tilde l_1-s_1+\tfrac12\tilde l_2\right)D^+\left(x-z-s_1-s_2+\tfrac12\tilde l_3\right)\\
&\qquad\qquad\times D^+(z+s_1+s_2+s_3-y)\\
&\qquad +D^+\left(-\tfrac12\tilde l_2-s_2-s_3\right)D^+\left(x-z+\tfrac12\tilde l_1\right)\\
&\qquad\qquad\times D^+\left(z+s_1+s_2-\tfrac12\tilde l_3-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_3-s_3\right)D^+\left(x-z+\tfrac12\tilde l_1\right)\\
&\qquad\qquad\times D^+\left(z+s_1-\tfrac12\tilde l_2-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_2-s_2+\tfrac12\tilde l_3\right)D^+\left(x-z+\tfrac12\tilde l_1\right)\\
&\qquad\qquad\times D^+(z+s_1+s_2+s_3-y)\Big\}\\
&\quad +\tau(z^0-x^0)\tau(x^0-y^0)\\
&\quad\times\Big\{\Big(D^+\left(-\tfrac12\tilde l_1-s_1-s_2-s_3\right)D^+\left(z+s_1-\tfrac12\tilde l_2-x\right)\\
&\qquad\qquad\times D^+\left(z+s_1+s_2-\tfrac12\tilde l_3-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_2-s_2-s_3\right)D^+\left(z-\tfrac12\tilde l_1-x\right)\\
&\qquad\qquad\times D^+\left(z+s_1+s_2-\tfrac12\tilde l_3-y\right)\\
&\qquad +D^+(-s_3-\tilde l_3)\,D^+\left(z-\tfrac12\tilde l_1-x\right)\\
&\qquad\qquad\times D^+\left(z+s_1-\tfrac12\tilde l_2-y\right)\\
&\qquad +D^+\left(-\tfrac12\tilde l_1-s_1-s_2+\tfrac12\tilde l_3\right)D^+\left(z+s_1-\tfrac12\tilde l_2-x\right)\\
&\qquad\qquad\times D^+(z+s_1+s_2+s_3-y)\\
&\qquad +D^+\left(-\tfrac12\tilde l_2-s_2+\tfrac12\tilde l_3\right)D^+\left(z-\tfrac12\tilde l_1-x\right)\\
&\qquad\qquad\times D^+(z+s_1+s_2+s_3-y)
\end{aligned}
$$

$$
\begin{aligned}
&+D^+\left(-\tfrac{1}{2}\tilde{l}_1-s_1+\tfrac{1}{2}\tilde{l}_2\right)D^+\left(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3-x\right)\\
&\quad\times D^+(z+s_1+s_2+s_3-y))\\
&+(x\leftrightarrow y)\Big\}\Big)+(x\leftrightarrow y)\Big\}\,. \qquad (17)
\end{aligned}
$$

We have to take the connected part $G^{\mathrm{con}}(x,y)$ only. Inserting (14) we can perform the s_i-integrations, which result in δ-distributions in l_i, so that the l_i-integration can be performed as well. The result has a remarkably compact form:

$$
\begin{aligned}
&G^{\mathrm{con}}(x,y)\\
&=\frac{g}{12}\int \mathrm{d}^4z\int\frac{\mathrm{d}^3k_1}{(2\pi)^3 2\omega_{k_1}}\int\frac{\mathrm{d}^3k_2}{(2\pi)^3 2\omega_{k_2}}\cos\left(\tfrac{1}{2}k_1^+\tilde{k}_2^+\right)\\
&\times\Big(\tau(x^0-y^0)\tau(y^0-z^0)\mathrm{e}^{-\mathrm{i}k_1^+(x-z)}\mathrm{e}^{-\mathrm{i}k_2^+(y-z)}\mathcal{I}^{++}(k_1^+,k_2^+)\\
&\quad+\tau(y^0-x^0)\tau(x^0-z^0)\mathrm{e}^{-\mathrm{i}k_1^+(x-z)}\mathrm{e}^{-\mathrm{i}k_2^+(y-z)}\mathcal{I}^{++}(k_1^+,k_2^+)\\
&\quad+\tau(x^0-z^0)\tau(z^0-y^0)\mathrm{e}^{-\mathrm{i}k_1^+(x-z)}\mathrm{e}^{-\mathrm{i}k_2^+(z-y)}\mathcal{I}^{+-}(k_1^+,k_2^+)\\
&\quad+\tau(y^0-z^0)\tau(z^0-x^0)\mathrm{e}^{-\mathrm{i}k_1^+(z-x)}\mathrm{e}^{-\mathrm{i}k_2^+(y-z)}\mathcal{I}^{-+}(k_1^+,k_2^+)\\
&\quad+\tau(z^0-x^0)\tau(x^0-y^0)\mathrm{e}^{-\mathrm{i}k_1^+(z-x)}\mathrm{e}^{-\mathrm{i}k_2^+(z-y)}\mathcal{I}^{--}(k_1^+,k_2^+)\\
&\quad+\tau(z^0-y^0)\tau(y^0-x^0)\mathrm{e}^{-\mathrm{i}k_1^+(z-x)}\mathrm{e}^{-\mathrm{i}k_2^+(z-y)}\\
&\qquad\times\mathcal{I}^{--}(k_1^+,k_2^+)\Big), \qquad (18)
\end{aligned}
$$

where $(\tilde{k}^+)^\nu\equiv(k^+)_\mu\theta^{\mu\nu}$ and

$$
\begin{aligned}
&\mathcal{I}^{\kappa\lambda}(k_1^+,k_2^+) \qquad (19)\\
&=\int\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\left(3+\mathrm{e}^{\mathrm{i}\kappa k_1^+\tilde{k}^++\mathrm{i}\lambda k_2^+\tilde{k}^+}+\mathrm{e}^{\mathrm{i}\kappa k_1^+\tilde{k}^+}+\mathrm{e}^{\mathrm{i}\lambda k_2^+\tilde{k}^+}\right),
\end{aligned}
$$

for $\kappa,\lambda=\pm 1$.

Next we pass to the Fourier-transformed Green's function

$$
G^{\mathrm{con}}(p,q)=\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{e}^{\mathrm{i}px+\mathrm{i}qy}\,G^{\mathrm{con}}(x,y)\,. \qquad (20)
$$

We insert the identity (use the residue theorem)

$$
\tau(x^0-y^0)=\lim_{\delta\to 0}\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t\frac{\mathrm{e}^{-\mathrm{i}t(x^0-y^0)}}{t+\mathrm{i}\delta} \qquad (21)
$$

and perform the integrations over x,y,z. The result is a host of δ-distributions, which allow us to integrate over $\vec{k}_1,\vec{k}_2,t_1,t_2$:

$$
\begin{aligned}
&G^{\mathrm{con}}(p,q)\\
&=\lim_{\delta_1,\delta_2\to 0}\frac{g}{12}\left(\frac{\mathrm{i}}{2\pi}\right)^2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\int_{-\infty}^{\infty}\frac{\mathrm{d}t_1}{t_1+\mathrm{i}\delta_1}\\
&\quad\times\int_{-\infty}^{\infty}\frac{\mathrm{d}t_2}{t_2+\mathrm{i}\delta_2}\int\frac{\mathrm{d}^3k_1}{(2\pi)^3 2\omega_{k_1}}\\
&\quad\times\int\frac{\mathrm{d}^3k_2}{(2\pi)^3 2\omega_{k_2}}\cos\left(\tfrac{1}{2}k_1^+\tilde{k}_2^+\right)\\
&\quad\times\Big(\mathrm{e}^{\mathrm{i}\{x^0(p_0-t_1-\omega_{k_1})+y^0(q_0+t_1-t_2-\omega_{k_2})+z^0(t_2+\omega_{k_1}+\omega_{k_2})\}}\\
&\qquad\times\mathrm{e}^{\mathrm{i}\{\vec{x}(\vec{k}_1-\vec{p})+\vec{y}(\vec{k}_2-\vec{q})-\vec{z}(\vec{k}_1+\vec{k}_2)\}}\mathcal{I}^{++}(k_1^+,k_2^+)\\
&\quad+\mathrm{e}^{\mathrm{i}\{x^0(p_0+t_1-t_2-\omega_{k_1})+y^0(q_0-t_1-\omega_{k_2})+z^0(t_2+\omega_{k_1}+\omega_{k_2})\}}\\
&\qquad\times\mathrm{e}^{\mathrm{i}\{\vec{x}(\vec{k}_1-\vec{p})+\vec{y}(\vec{k}_2-\vec{q})-\vec{z}(\vec{k}_1+\vec{k}_2)\}}\mathcal{I}^{++}(k_1^+,k_2^+)\\
&\quad+\mathrm{e}^{\mathrm{i}\{x^0(p_0-t_1-\omega_{k_1})+y^0(q_0+t_2+\omega_{k_2})+z^0(t_1-t_2+\omega_{k_1}-\omega_{k_2})\}}\\
&\qquad\times\mathrm{e}^{\mathrm{i}\{\vec{x}(\vec{k}_1-\vec{p})-\vec{y}(\vec{k}_2+\vec{q})+\vec{z}(\vec{k}_2-\vec{k}_1)\}}\mathcal{I}^{+-}(k_1^+,k_2^+)\\
&\quad+\mathrm{e}^{\mathrm{i}\{x^0(p_0+t_2+\omega_{k_1})+y^0(q_0-t_1-\omega_{k_2})+z^0(t_1-t_2-\omega_{k_1}+\omega_{k_2})\}}\\
&\qquad\times\mathrm{e}^{\mathrm{i}\{-\vec{x}(\vec{k}_1+\vec{p})+\vec{y}(\vec{k}_2-\vec{q})+\vec{z}(\vec{k}_1-\vec{k}_2)\}}\mathcal{I}^{-+}(k_1^+,k_2^+)\\
&\quad+\mathrm{e}^{\mathrm{i}\{x^0(p_0+t_1-t_2+\omega_{k_1})+y^0(q_0+t_2+\omega_{k_2})-z^0(t_1+\omega_{k_1}+\omega_{k_2})\}}\\
&\qquad\times\mathrm{e}^{\mathrm{i}\{-\vec{x}(\vec{k}_1+\vec{p})-\vec{y}(\vec{k}_2+\vec{q})+\vec{z}(\vec{k}_1+\vec{k}_2)\}}\mathcal{I}^{--}(k_1^+,k_2^+)\\
&\quad+\mathrm{e}^{\mathrm{i}\{x^0(p_0+t_2+\omega_{k_1})+y^0(q_0+t_1-t_2+\omega_{k_2})-z^0(t_1+\omega_{k_1}+\omega_{k_2})\}}\\
&\qquad\times\mathrm{e}^{\mathrm{i}\{-\vec{x}(\vec{k}_1+\vec{p})-\vec{y}(\vec{k}_2+\vec{q})+\vec{z}(\vec{k}_1+\vec{k}_2)\}}\mathcal{I}^{--}(k_1^+,k_2^+)\Big)\\
&=\lim_{\delta_1,\delta_2\to 0}\frac{g}{12}(2\pi)^4\delta(p+q)\\
&\times\Bigg(\frac{1}{p_0-\omega_p+\mathrm{i}\delta_1}\,\frac{1}{\omega_p+\omega_q-\mathrm{i}\delta_2}\,\frac{\cos\left(\frac{1}{2}p^+\tilde{q}^+\right)}{4\omega_p\omega_q}\mathcal{I}^{++}(p^+,q^+)\\
&\quad+\frac{1}{q_0-\omega_q+\mathrm{i}\delta_1}\,\frac{1}{\omega_p+\omega_q-\mathrm{i}\delta_2}\,\frac{\cos\left(\frac{1}{2}p^+\tilde{q}^+\right)}{4\omega_p\omega_q}\mathcal{I}^{++}(p^+,q^+)\\
&\quad+\frac{1}{p_0-\omega_p+\mathrm{i}\delta_1}\,\frac{1}{q_0+\omega_q-\mathrm{i}\delta_2}\,\frac{\cos\left(\frac{1}{2}p^+(-\tilde{q})^+\right)}{4\omega_p\omega_q}\\
&\qquad\times\mathcal{I}^{+-}(p^+,(-q)^+)\\
&\quad+\frac{1}{q_0-\omega_q+\mathrm{i}\delta_1}\,\frac{1}{p_0+\omega_p-\mathrm{i}\delta_2}\,\frac{\cos\left(\frac{1}{2}(-p)^+\tilde{q}^+\right)}{4\omega_p\omega_q}\\
&\qquad\times\mathcal{I}^{-+}((-p)^+,q^+)\\
&\quad+\frac{1}{\omega_p+\omega_q-\mathrm{i}\delta_1}\,\frac{1}{-q_0-\omega_q+\mathrm{i}\delta_2}\,\frac{\cos\left(\frac{1}{2}(-p)^+(-\tilde{q})^+\right)}{4\omega_p\omega_q}\\
&\qquad\times\mathcal{I}^{--}((-p)^+,(-q)^+)\\
&\quad+\frac{1}{\omega_p+\omega_q-\mathrm{i}\delta_1}\,\frac{1}{-p_0-\omega_p+\mathrm{i}\delta_2}\,\frac{\cos\left(\frac{1}{2}(-p)^+(-\tilde{q})^+\right)}{4\omega_p\omega_q}\\
&\qquad\times\mathcal{I}^{--}((-p)^+,(-q)^+)\Bigg)\,. \qquad (22)
\end{aligned}
$$

Note the appearance of $\delta(p+q)$ implementing conservation of the four-momentum (translation invariance). We have used $\omega_{\pm k}=\omega_k$.

Following [10] we amputate the external legs by multiplying (22) by the inverse propagators $-\mathrm{i}(p_0^2-\omega_p^2)$ and $-\mathrm{i}(q_0^2-\omega_q^2)$. Using $(\pm k)^+=\pm k^\pm$, in particular the identity

$$
\begin{aligned}
&\mathcal{I}^{\pm\pm}((\pm p)^+,(\pm q)^+)\\
&=\int\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\left(3+\mathrm{e}^{\mathrm{i}p^\pm\tilde{k}^++\mathrm{i}q^\pm\tilde{k}^+}+\mathrm{e}^{\mathrm{i}p^\pm\tilde{k}^+}+\mathrm{e}^{\mathrm{i}q^\pm\tilde{k}^+}\right)\\
&\equiv\mathcal{I}(p^\pm,q^\pm)\,, \qquad (23)
\end{aligned}
$$

we obtain

$$
(2\pi)^4\delta(p+q)\mathit{\Gamma}(p,q)
$$

$$
\begin{aligned}
&= -(p_0^2-\omega_p^2)(q_0^2-\omega_q^2)G(p,q) \\
&= -\lim_{\delta_1,\delta_2\to 0}\frac{g}{12}(2\pi)^4\delta(p+q)\,(p_0^2-\omega_p^2)(q_0^2-\omega_q^2) \\
&\quad\times\Bigg(\frac{1}{p_0-\omega_p+\mathrm{i}\delta_1}\,\frac{1}{\omega_p+\omega_q-\mathrm{i}\delta_2}\,\frac{\cos\left(\frac12 p^+\tilde q^+\right)}{4\omega_p\omega_q}\mathcal{I}(p^+,q^+) \\
&\quad+\frac{1}{q_0-\omega_q+\mathrm{i}\delta_1}\,\frac{1}{\omega_p+\omega_q-\mathrm{i}\delta_2}\,\frac{\cos\left(\frac12 p^+\tilde q^+\right)}{4\omega_p\omega_q}\mathcal{I}(p^+,q^+) \\
&\quad+\frac{1}{p_0-\omega_p+\mathrm{i}\delta_1}\,\frac{1}{q_0+\omega_q-\mathrm{i}\delta_2}\,\frac{\cos\left(\frac12 p^+\tilde q^-\right)}{4\omega_p\omega_q}\mathcal{I}(p^+,q^-) \\
&\quad+\frac{1}{q_0-\omega_q+\mathrm{i}\delta_1}\,\frac{1}{p_0+\omega_p-\mathrm{i}\delta_2}\,\frac{\cos\left(\frac12 p^-\tilde q^+\right)}{4\omega_p\omega_q}\mathcal{I}(p^-,q^+) \\
&\quad+\frac{1}{\omega_p+\omega_q-\mathrm{i}\delta_1}\,\frac{1}{-q_0-\omega_q+\mathrm{i}\delta_2}\,\frac{\cos\left(\frac12 p^-\tilde q^-\right)}{4\omega_p\omega_q}\mathcal{I}(p^-,q^-) \\
&\quad+\frac{1}{\omega_p+\omega_q-\mathrm{i}\delta_1}\,\frac{1}{-p_0-\omega_p+\mathrm{i}\delta_2} \\
&\qquad\times\frac{\cos\left(\frac12 p^-\tilde q^-\right)}{4\omega_p\omega_q}\mathcal{I}(p^-,q^-)\Bigg)\,. \qquad (24)
\end{aligned}
$$

Taking on-shell external momenta $p_0=\omega_p$ and $q_0=-\omega_q$ there survives a single term (the third one):

$$
\begin{aligned}
\Gamma(p^+,q^-) &= \lim_{p_0\to\omega_p\,,\,q_0\to-\omega_q}\Gamma(p,q) \\
&= \frac{g}{12}\cos\left(\tfrac12 p^+\tilde q^-\right)\mathcal{I}(p^+,q^-) \qquad (25)\\
&= \frac{g}{12}\int\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\Big(4+2\cos(k^+\tilde p^+)\Big)\,.
\end{aligned}
$$

In the last line we have used momentum conservation $p^+=-q^-$ and the skew-symmetry of θ. The remaining integral over $\vec k$ consists of a planar θ-independent part and a non-planar θ-dependent part (the cosine). The planar part coincides (up to a factor $\frac23$) with the commutative result; it is divergent and is to be renormalized as usual by multiplicative renormalization (or better completely removed by normal ordering).

To compute the non-planar part, first note that

$$
\begin{aligned}
\cos(k^+\tilde p^+) &= \cos\left(\omega_k\tilde p_0-\vec k\vec{\tilde p}\right) \qquad (26)\\
&= \cos\left(\omega_k\tilde p_0\right)\cos(\vec k\vec{\tilde p})+\sin\left(\omega_k\tilde p_0\right)\sin(\vec k\vec{\tilde p})\,,
\end{aligned}
$$

where $\tilde p_0 := (\tilde p^+)_0$ and $\vec{\tilde p}=\overrightarrow{\tilde p^+}$. The uneven sine-term will drop under the integral. Using the residue theorem we have

$$
\frac{\mathrm{e}^{\mathrm{i}\omega_k\tilde p_0}}{2\omega_k}
=\begin{cases}
\displaystyle\lim_{\epsilon\to0}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}k_0\,\frac{\mathrm{e}^{-\mathrm{i}k_0\tilde p_0}}{(k_0+\omega_k+\mathrm{i}\epsilon)(k_0-\omega_k-\mathrm{i}\epsilon)} & \text{for } \tilde p_0>0\,,\\[2ex]
\displaystyle\lim_{\epsilon\to0}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}k_0\,\frac{-\mathrm{e}^{-\mathrm{i}k_0\tilde p_0}}{(k_0+\omega_k-\mathrm{i}\epsilon)(k_0-\omega_k+\mathrm{i}\epsilon)} & \text{for } \tilde p_0<0\,,
\end{cases} \qquad (27)
$$

$$
\frac{\mathrm{e}^{-\mathrm{i}\omega_k\tilde p_0}}{2\omega_k}
=\begin{cases}
\displaystyle\lim_{\epsilon\to0}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}k_0\,\frac{-\mathrm{e}^{-\mathrm{i}k_0\tilde p_0}}{(k_0+\omega_k-\mathrm{i}\epsilon)(k_0-\omega_k+\mathrm{i}\epsilon)} & \text{for } \tilde p_0>0\,,\\[2ex]
\displaystyle\lim_{\epsilon\to0}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}k_0\,\frac{\mathrm{e}^{-\mathrm{i}k_0\tilde p_0}}{(k_0+\omega_k+\mathrm{i}\epsilon)(k_0-\omega_k-\mathrm{i}\epsilon)} & \text{for } \tilde p_0<0\,.
\end{cases} \qquad (28)
$$

Inserting (26), (27) and (28) into (25) we obtain for the non-planar graph

$$
\begin{aligned}
&\Gamma_{\text{non-planar}}(p^+,q^-) \\
&\quad\equiv \frac{g}{6}\int\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\cos(k^+\tilde p^+) \qquad (29)\\
&\quad= \lim_{\epsilon\to0}\frac{g}{6}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\Re\left(\frac{\mathrm{i}}{k_0^2-(\vec k^2+m^2)+\mathrm{i}\epsilon}\right)\mathrm{e}^{-\mathrm{i}k\tilde p^+}\,,
\end{aligned}
$$

independent of the sign of $\tilde p_0$. The result (29) can obviously be obtained by Feynman rules, with the prescription that in non-planar tadpoles the propagator to use is *the real part of the Feynman propagator*. That real part is arithmetic mean of causal and acausal propagators. The observed acausality is no surprise, because according to (10) the interaction time-ordering T_{I} explicitly violates causality. As we shall see in Sect. 4, the just given Feynman rule is true for tadpole lines only.

Apart from taking the real part, the evaluation of (29) coincides with the computation in the "naïve" Feynman graph approach. Let us nevertheless repeat the steps. We employ Zimmermann's ϵ-trick

$$
\begin{aligned}
\frac{1}{k^2-m^2+\mathrm{i}\epsilon} &\mapsto \frac{1}{k_0^2+\omega_k^2(\mathrm{i}\epsilon-1)} \\
&= \frac{\epsilon'-\mathrm{i}}{(\epsilon'-\mathrm{i})k_0^2+\omega_k^2(\epsilon-\epsilon'+\mathrm{i}+\mathrm{i}\epsilon\epsilon')}\,, \qquad (30)
\end{aligned}
$$

the denominator of which has for $\epsilon'<\epsilon$ a positive real part, which allows us to introduce a Schwinger parameter:

$$
\begin{aligned}
&\Gamma_{\text{non-planar}}(p^+,q^-) \\
&= \Re\Bigg(\lim_{\epsilon\to0,\epsilon'<\epsilon}\frac{\mathrm{i}g}{6}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^\infty\mathrm{d}\alpha\,(\epsilon'-\mathrm{i}) \\
&\qquad\times\mathrm{e}^{-\alpha\{(\epsilon'-\mathrm{i})k_0^2+(\vec k^2+m^2)(\epsilon-\epsilon'+\mathrm{i}+\mathrm{i}\epsilon\epsilon')\}-\mathrm{i}k_0\tilde p_0+\mathrm{i}\vec k\vec{\tilde p}}\Bigg) \\
&= \Re\Bigg(\lim_{\epsilon\to0,\epsilon'<\epsilon}\frac{\mathrm{i}g}{6(4\pi)^2}\frac{(\epsilon'-\mathrm{i})^{\frac12}}{(\epsilon-\epsilon'+\mathrm{i}+\mathrm{i}\epsilon\epsilon')^{\frac32}} \\
&\qquad\times\int_0^\infty\frac{\mathrm{d}\alpha}{\alpha^2}\,\mathrm{e}^{-\frac{\tilde p_0^2}{4\alpha(\epsilon'-\mathrm{i})}-\frac{\vec{\tilde p}^2}{4\alpha(\epsilon-\epsilon'+\mathrm{i}+\mathrm{i}\epsilon\epsilon')}-\alpha m^2(\epsilon-\epsilon'+\mathrm{i}+\mathrm{i}\epsilon\epsilon')}\Bigg) \\
&= \Re\Bigg(\lim_{\epsilon\to0}\frac{2\mathrm{i}g}{3(4\pi)^2}\frac{1}{(\mathrm{i}\epsilon-1)^{\frac32}}\sqrt{\frac{m^2(\mathrm{i}\epsilon-1)}{\tilde p_0^2+\frac{\vec{\tilde p}^2}{(\mathrm{i}\epsilon-1)}}} \\
&\qquad\times K_1\Big(\sqrt{m^2(\vec{\tilde p}^2+(\mathrm{i}\epsilon-1)\tilde p_0^2)}\Big)\Bigg)
\end{aligned}
$$

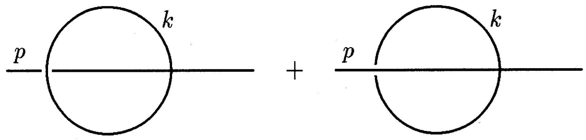
$$= -\Re\left(\frac{2g}{3(4\pi)^2}\sqrt{-\frac{m^2}{\tilde{p}^2}}K_1\big(\sqrt{-\tilde{p}^2m^2}\big)\right) . \tag{31}$$

We have used $\int_0^\infty \frac{\mathrm{d}\alpha}{\alpha^2}\exp(-u\alpha-v/(4\alpha)) = 4\sqrt{\frac{u}{v}}K_1(\sqrt{uv})$ for $\Re u > 0$ and $\Re v > 0$.

In the particular case where the external momentum p is put on-shell, we have

$$-\tilde{p}^2 = \vec{\tilde{p}}^2 - \tilde{p}_0^2 = (\theta_{i0}\sqrt{\vec{p}^2+m^2} + \theta_{ij}p^j)^2 - (\theta_{0j}p^j)^2 \geq 0 , \tag{32}$$

because $\tilde{p}^\mu$ has to be space-like or null as a vector which is orthogonal to the time-like vector p^μ. Thus, the projection onto the real part in (31) is superfluous, and (31) agrees exactly with the naïve Feynman rule computation of the sum of graphs



. (33)

However, if these graphs appear as subgraphs in a bigger graph, the momentum p will be the off-shell momentum through a propagator, and the projection to the real part makes a difference.

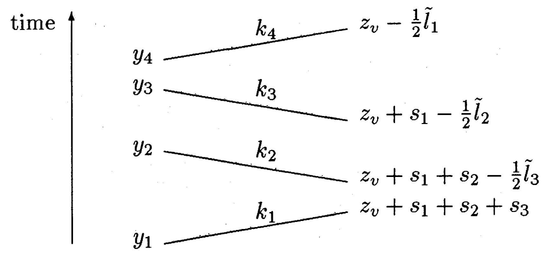
4 The general case

The graph we have computed (for off-shell external momenta!) is very often made responsible for the so-called UV/IR mixing. In fact the situation is more complex, as it is very well described in [4]. The ultimate goal must be to derive the power-counting theorem for interaction-point time-ordered perturbation theory (for non-commutative space and time). In a first step one has to derive graphical rules to assign an integral to a given graph.

Let us therefore consider the momentum integral for a general Feynman graph for a non-commutative ϕ^4 theory. A given connected contribution to the E-point function at order V in the coupling constant has after performing the Wick contractions, insertion of the D^+ according to (14), integration over s_i and l_i appearing in (6) and insertion of step functions (21), the form

$$\begin{aligned}
&G(x_1,\dots,x_E)\\
&= \lim_{\epsilon\to 0}\int\prod_{v=1}^{V}\frac{g\,\mathrm{d}^4z_v}{4!}\int\prod_{s=1}^{E+V-1}\frac{\mathrm{i}\,\mathrm{d}t_s}{(2\pi)(t_s+\mathrm{i}\epsilon)}\\
&\quad\times\int\prod_{e=1}^{E}\frac{\mathrm{d}^3p_e}{(2\pi)^3 2\omega_{p_e}}\int\prod_{i=1}^{I}\frac{\mathrm{d}^3k_i}{(2\pi)^3 2\omega_{k_i}}\\
&\quad\times\exp\left(-\mathrm{i}\sum_{v=1}^{V}\sum_{s=1}^{E+V-1}T_{vs}z_v^0t_s - \mathrm{i}\sum_{e=1}^{E}\sum_{s=1}^{E+V-1}T_{es}x_e^0t_s\right)\\
&\quad\times\exp\left(-\mathrm{i}\sum_{v=1}^{V}z_v\left(\sum_{i=1}^{I}J_{vi}k_i^+ + \sum_{e=1}^{E}J_{ve}p_e^+\right)\right)\\
&\quad\times\exp\left(-\mathrm{i}\sum_{e=1}^{E}\sigma_e p_e^+ x_e\right)\\
&\quad\times\exp\left(\mathrm{i}\theta^{\mu\nu}\left(\sum_{i,j=1}^{I}I_{ij}k_{i,\mu}^+k_{j,\nu}^+ + \sum_{i=1}^{I}\sum_{e=1}^{E}I_{ie}k_{i,\mu}^+p_{e,\nu}^+\right.\right.\\
&\qquad\left.\left.+\sum_{e,f=1}^{E}I_{ef}p_{e,\mu}^+p_{f,\nu}^+\right)\right) .
\end{aligned}\tag{34}$$

There are $E+V-1$ step functions according to the time differences of the E external points x_e and the V interaction points z_v. For each s there are two non-vanishing T_{*s}, where these two indices $*$ are either two indices e, one index e and one index v, or two indices v. The T_{*s} for which the vertex $*$ (z_v or x_e) is later equals $+1$, the other one -1. This gives the second line in (34). An external point x_e is linked via the external line with momentum p_e to exactly one vertex z_v, i.e. for given e there is a single non-vanishing J_{ve}. For our ϕ^4 theory there are $I = 2V - \frac{1}{2}E$ internal lines (E is even) with momentum k_i which link a vertex z_v to another vertex $z_{v'}$. Thus, if $v \neq v'$ (no tadpoles) for given i there are two non-vanishing J_{vi}, whereas for $v = v'$ we have $J_{vi}k_i^+ \equiv 0$. We orient the internal and external lines forward in time. Then, the incidence matrices J_{vi}, J_{ve} equal -1 if the line leaves v and $+1$ if the line arrives at v. Similarly, $\sigma_e = -1$ if the line e leaves x_e and $\sigma_e = +1$ if the line e arrives at x_e. The matrices I_{ij}, I_{ie}, I_{ef} are the intersection matrices [13,4], which instead of the Euclidian rosette construction are in IPTO obtained as follows: According to the definition (6) of the $\star$-product, write at each vertex v the four fields in (6) as a time-sequence where $z_v-\frac{1}{2}\tilde{l}_1$ is the latest point and $z_v+s_1+s_2+s_3$ the earliest point[2], irrespective of the *actual time-order* of these four points. Connect these points with vertices y_1, y_2, y_3, v_4 according to the following picture:



. (35)

The phase factor produced by the s_n and l_n variables is then given by

$$\begin{aligned}
&\int\prod_{n=1}^{3}\left(\mathrm{d}^4s_n\frac{\mathrm{d}^4l_n}{(2\pi)^4}\;\exp(\mathrm{i}s_nl_n)\right)\\
&\quad\times\exp\Big(-\mathrm{i}k_1^+(s_1+s_2+s_3)J_{v1} - \mathrm{i}k_2^+\left(s_1+s_2-\tfrac{1}{2}\tilde{l}_3\right)J_{v2}\\
&\qquad-\mathrm{i}k_3^+\left(s_1-\tfrac{1}{2}\tilde{l}_2\right)J_{v3} - \mathrm{i}k_4^+\left(-\tfrac{1}{2}\tilde{l}_1\right)J_{v4}\Big)
\end{aligned}$$

[2] By the way, this defines the time-orientation of tadpole lines

$$= \exp\left(\frac{\mathrm{i}}{2}\theta^{\mu\nu}\sum_{j=2}^{4}\sum_{i=1}^{j-1} k^+_{i,\mu}J_{vi}k^+_{j,\nu}J_{vj}\right)$$

$$\equiv \exp\left(\frac{\mathrm{i}}{2}\theta^{\mu\nu}\sum_{i,j=1}^{4}\tau^v_{ij}k^+_{i,\mu}J_{vi}k^+_{j,\nu}J_{vj}\right) . \tag{36}$$

We have to define $\tau^v_{ij} = +1$ if the line i is connected to an "earlier" field ϕ in the vertex v than the line j, otherwise $\tau^v_{ij} = 0$. Summing over all vertices and distinguishing external and internal lines, we are led to the following identification in (34):

$$\begin{aligned} I_{ij} &= \tfrac{1}{2}\sum_{v=1}^{V}\tau^v_{ij}J_{vi}J_{vj} , \\ I_{ie} &= \tfrac{1}{2}\sum_{v=1}^{V}\left(\tau^v_{ie} - \tau^v_{ei}\right)J_{vi}J_{ve} , \\ I_{ef} &= \tfrac{1}{2}\sum_{v=1}^{V}\tau^v_{ef}J_{ve}J_{vf} . \end{aligned} \tag{37}$$

Once more we notice the enormous computational advantage of using the $\star$-product in the form (4).

We perform the Fourier transformation $\int\prod_{e=1}^{E}\big(\mathrm{d}^4x_e \exp(\mathrm{i}q_e x_e)\big)$ of (34) to external momentum variables q as well as the z_v-integrations:

$$\begin{aligned} G(q_1,\dots,q_E) &= \lim_{\epsilon\to 0}\frac{g^V}{(4!)^V}\prod_{e=1}^{E}\frac{1}{2\omega_{q_e}} \\ &\times\int\prod_{i=1}^{I}\frac{\mathrm{d}^3k_i}{(2\pi)^3 2\omega_{k_i}}\int\prod_{s=1}^{E+V-1}\frac{\mathrm{i}\,\mathrm{d}t_s}{(2\pi)(t_s+\mathrm{i}\epsilon)} \\ &\times\prod_{v=1}^{V}(2\pi)^3\delta^3\left(\sum_{i=1}^{I}J_{vi}\vec{k}_i + \sum_{e=1}^{E}J_{ve}\sigma_e\vec{q}_e\right) \\ &\times\prod_{e=1}^{E}(2\pi)\delta\left(q^0_e - \sigma_e\omega_{q_e} - \sum_{s=1}^{E+V-1}T_{es}t_s\right) \\ &\times\prod_{v=1}^{V}(2\pi)\delta\left(\sum_{i=1}^{I}J_{vi}\omega_{k_i} + \sum_{e=1}^{E}J_{ve}\omega_{q_e} + \sum_{s=1}^{E+V-1}T_{vs}t_s\right) \\ &\times\exp\left(\mathrm{i}\theta^{\mu\nu}\left(\sum_{i,j=1}^{I}I_{ij}k^+_{i,\mu}k^+_{j,\nu} + \sum_{i=1}^{I}\sum_{e=1}^{E}I_{ie}\sigma_e k^+_{i,\mu}q^{\sigma_e}_{e,\nu}\right.\right. \\ &\left.\left.+\sum_{e,f=1}^{E}I_{ef}\sigma_e\sigma_f q^{\sigma_e}_{e,\mu}q^{\sigma_f}_{f,\nu}\right)\right) . \end{aligned} \tag{38}$$

The vectors $\vec{q}_e$ are always outgoing from internal vertices. There are now $E{+}V$ time-component δ-functions involving the $E{+}V{-}1$ integration variables t_s, after integration over which there is one remaining δ-function for the energy conservation $\delta\left(\sum_{e=1}^{E}q^0_e\right)$. We multiply (38) by the inverse propagators $\prod_{e=1}^{E}(-\mathrm{i})((q^0_e)^2 - \omega^2_{q_e})$, remove $(2\pi)^4\delta^4\left(\sum_{e=1}^{E}q_e\right)$ by convention and put $q^0_e = \sigma_e\omega_{q_e}$. There is a non-vanishing contribution only if the external vertices x_e are either before or after the internal vertices z_i. Defining a time-order of vertices $v' < v$ if $z^0_{v'} < z^0_v$ we finally get

$$\begin{aligned} &\Gamma(q_1^{\sigma_1},\dots,q_E^{\sigma_E}) \\ &= \lim_{\epsilon\to 0}\frac{g^V}{(4!)^V}\int\prod_{i=1}^{I}\frac{\mathrm{d}^3k_i}{(2\pi)^3 2\omega_{k_i}} \\ &\quad\times\prod_{v=1}^{V-1}\frac{\mathrm{i}(2\pi)^3\delta^3\left(\sum_{i=1}^{I}J_{vi}\vec{k}_i + \sum_{e=1}^{E}J_{ve}\sigma_e\vec{q}_e\right)}{\sum_{v'\le v}\left(\sum_{i=1}^{I}J_{v'i}\omega_{k_i} + \sum_{e=1}^{E}J_{v'e}\omega_{q_e}\right) + \mathrm{i}\epsilon} \\ &\quad\times\exp\left(\mathrm{i}\theta^{\mu\nu}\left(\sum_{i,j=1}^{I}I_{ij}k^+_{i,\mu}k^+_{j,\nu} + \sum_{i=1}^{I}\sum_{e=1}^{E}I_{ie}\sigma_e k^+_{i,\mu}q^{\sigma_e}_{e,\nu}\right.\right. \\ &\quad\left.\left.+\sum_{e,f=1}^{E}I_{ef}\sigma_e\sigma_f q^{\sigma_e}_{e,\mu}q^{\sigma_f}_{f,\nu}\right)\right) . \end{aligned} \tag{39}$$

The vertex which is missing in the product over v is the latest one. There remain $I{-}V{+}1{=}L$ momentum integrations to perform, where L is the number of loops. The integral (39) corresponds to a particular graph with E external and V internal vertices which all have different dates. The internal vertices are composed of four different points according to the four fields building the vertex, with the time-interval within a vertex smaller than the time-distance to the neighbored vertices. Any external vertex is a single point which is either later or earlier than all points in internal vertices. A graph is the connection of each two of these $4V{+}E$ points by a line which is oriented forward in time, such that at each point we find exactly one end of a line. We assign to this graph the integral (39) according to the incidence matrices, which also enter in (37). Finally, one has to sum over all different graphs. Note that a given graph does not have any symmetry because the four points in the vertices have clearly distinguished dates. The Feynman rule (39) is easily generalized to other than ϕ^4 theories. Equation (39) is the analytic expression of the Feynman rules listed in [10], apart from a disagreement in the symmetry factor.

We now see that the graph we have computed was very special. Because of $V = 1$ the denominator in (39) was absent so that the integration over the propagator momentum k_1 was identical to the naïve Feynman graph computation. This remains true for all tadpole lines i, because for them $J_{vi}k^+_i = 0$ for all v. For internal lines connecting points in different vertices we need new techniques to perform the integrations.

5 Summary

As a warm-up for the general treatment we have computed the one-loop two-point function for a ϕ^4 theory on non-commutative space and time in the framework of "interaction-point time-ordered perturbation theory". The calculation is based on free fields (on the mass shell), but at the end the loop momenta become general four-momenta. Our final result (for that graph) agrees with a Feynman graph computation, provided that *one assigns to the internal line the real part of the Feynman propagator*. This can be understood as the inclusion of acausal processes in the S-matrix, because IPTO explicitly violates causality. One may speculate that the true time-ordering of the $\star$-product (9) will produce the naïve Feynman rules involving the standard causal Feynman propagator in non-planar graphs. This approach was shown to violate unitarity of the S-matrix. We have thus to decide whether we prefer to give up (micro-) causality or unitarity in non-commutative field theories[3].

Next we have derived the Feynman rules (39) for general Green's functions. Power-counting tells us that (39) is expected to diverge if there are subgraphs with $E \leq 4$ external lines. If there are non-planar divergent graphs, it is not possible to absorb the divergences by local (hence planar) counterterms. One has therefore to analyze whether the oscillating phases render the power-counting divergent integral finite. This requires one to develop techniques for the computation of (39) in analogy to the treatment of the Euclidian case in [4]. Of urgent interest are the evaluations of the two-loop two-point function and the one-loop four-point function.

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Appendix

A Conventions for Fock space and propagators

To fix our notation and for convenience we list our conventions for free fields and propagators $D^{\pm}(x-y)$ and $\Delta_{\mathrm{F}}(x-y)$.

The free fields (solutions of the homogeneous Klein–Gordon equation) are mode-decomposed into negative (ϕ^+) and positive (ϕ^-) frequency parts $\phi(x) = \phi^+(x) + \phi^-(x)$,

$$\begin{aligned} \phi^-(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}\, a_k^- \mathrm{e}^{-\mathrm{i}x_\mu k^{+\mu}} , \\ \phi^+(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}\, a_k^+ \mathrm{e}^{+\mathrm{i}x_\mu k^{+\mu}} , \end{aligned} \tag{40}$$

with the ladder operators a^-, a^+ obeying

$$a_k^-|0\rangle = 0 , \quad \langle 0|a_k^+ = 0 , \quad [a_p^-, a_q^+] = \delta^3(\vec{p}-\vec{q}) . \tag{41}$$

With these definitions we obtain for the two-point vacuum expectation values and the commutators of positive and negative frequency parts

$$\begin{aligned} \langle 0|\phi(x)\phi(y)|0\rangle &= [\phi^-(x), \phi^+(y)] = D^+(x-y) \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\, \mathrm{e}^{-\mathrm{i}(x-y)_\mu k^{+\mu}} , \\ \langle 0|\phi(y)\phi(x)|0\rangle &= -[\phi^+(x), \phi^-(y)] = D^-(x-y) \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\, \mathrm{e}^{\mathrm{i}(x-y)_\mu k^{+\mu}} , \end{aligned} \tag{42}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$ and $(k^{\pm})^\mu = (\pm\omega_k, \vec{k})^\mu$. For the Feynman propagator we hence find

$$\begin{aligned} \langle 0|T\big(\phi(x)\phi(y)\big)|0\rangle &= \Delta_{\mathrm{F}}(x-y) \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}(x-y)k}}{k^2 - m^2 + \mathrm{i}\varepsilon} , \end{aligned} \tag{43}$$

and for its complex conjugate

$$\begin{aligned} &\langle 0|\tau(y^0 - x^0)\phi(x)\phi(y) + \tau(x^0 - y^0)\phi(y)\phi(x)|0\rangle \\ &= \Delta_{\mathrm{F}}^*(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{-\mathrm{i}\mathrm{e}^{-\mathrm{i}(x-y)k}}{k^2 - m^2 - \mathrm{i}\varepsilon} . \end{aligned} \tag{44}$$

These propagators are solutions of the homogeneous and inhomogeneous wave equation, respectively:

$$\begin{aligned} (\partial_\mu\partial^\mu - m^2)_x D^{\pm}(x-y) &= 0 , \\ (\partial_\mu\partial^\mu - m^2)_x \Delta_{\mathrm{F}}(x-y) &= -\mathrm{i}\delta^4(x-y) . \end{aligned} \tag{45}$$

[3] Assuming space-time non-commutativity to be a model of quantum-gravitational background effects ($\theta \sim l^2_{\mathrm{Planck}}$), one can view this abandonment of causality in the $\star$-product as its breakdown at the Planck scale

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